Ques - Form an integral equation corresponding to the differential equation given by

$$\frac{d^2y}{dx^2} + xy = 1$$

$$y(0) = y'(0) = 0$$

Soln

The given differential equation is

$$\frac{d^2y}{dx^2} + xy = 1 \qquad \text{——— (1)}$$

with initial conditions

$$y(0) = y'(0) = 0 \qquad \text{——— (2)}$$

Let us suppose

$$\frac{d^2y}{dx^2} = u(x) \qquad \text{——— (3)}$$

Integrating equation (3) w.r.to x from 0 to x, we get

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x u(x)\,dx$$

$$\Rightarrow \frac{dy}{dx} - y'(0) = \int_0^x u(x)\,dx$$

$$\Rightarrow \frac{dy}{dx} = \int_0^x u(x)\,dx \qquad \text{—— (4)}$$

$$[\because y'(0) = 0]$$

$$\Rightarrow \frac{dy}{dx} = \int_0^x u(t)\,dt \qquad \text{—— (5)}$$

$$y(x) - y(0) = \int_0^x u(x)\,dx^2$$

$$\Rightarrow y(x) = \int_0^x (x-t)\,u(t)\,dt \qquad \text{—— (6)}$$

$$[\because y(0) = 0]$$

Now putting the values of $\frac{d^2y}{dx^2}$ and y from above equation, in the given equation, we get

$$u(x) + x\int_0^x (x-t)\,u(t)\,dt = 1$$

$$\Rightarrow u(x) = 1 - \int_0^x x\,(x-t)\,u(t)\,dt \qquad \text{—— (7)}$$

Which is the required integral equation.